

Relativistic meson spectroscopy in momentum space

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Abstract

In this paper a relativistic quantum theory introduced by de Groot and Ruijgrok in 1975, is used as a quark model in momentum space. The complete spectrum, with the exception of the selfconjugate light unflavoured mesons, is calculated. The potential used consists of an one-gluon-exchange (OGE) part and a confining part. For the confining part a relativistic generalization of the linear plus constant potential was used, which is well-defined in momentum space without introducing any singularities. For the OGE part several potentials were investigated. Retardations were included at all places. Using a fitting procedure involving 52 well-established mesons, best results were obtained for a potential consisting of a purely vector Richardson potential and a purely scalar confining potential. In this way a vector confining is entirely induced by the Richardson potential. Reasonable results were also obtained for a modified Richardson potential. Most meson masses, with the exception of the π , the K and the K_0^* , were found to be reasonably well described by the model.

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I. INTRODUCTION

In principle the properties of mesons and baryons should be correctly described by Quantum chromodynamics (QCD). However, apart from some lattice gauge calculations, this is practically impossible at the moment. As a replacement simple quark models, in which hadrons are viewed as bound states of constituent quarks, are quite successful (for a review see [1]). The simplest are the nonrelativistic (NR) ones. The potential used here, normally consists of a Coulomb term to account for the perturbative one gluon exchange (OGE), and a linear potential with possibly an additional constant to incorporate the nonperturbative confining. These models work very well for the heavier charmonia and bottonia. For the lighter mesons however, it is clear that relativistic corrections must be included. Roughly speaking, this can be achieved in two ways. The most direct way is to replace all NR expressions by their relativistic counterparts. Spin dependences, like spin-spin, spin-orbit and tensor couplings are included by hand. The second way, which, from a theoretical point of view is more consistent, is to start from a framework that is manifest Lorentz covariant from the outset. The most natural representation for such a framework, like the Bethe-Salpeter equations, is momentum space. Traditionally, however, and also because of the belief that it would be impossible to describe a confining potential in momentum space, the equations are normally transformed to configuration space. Since a few years [2–7], however, it has been realized, that there is no obstacle to define a confining potential in momentum space, even in the relativistic case. Therefore a growing interest has arisen to study quark models directly in this more favourable representation.

This will also be the subject of the present paper. The theory introduced by de Groot and Ruijgrok [8–12] will be used as a model to calculate the masses of all known mesons, with the exception of the selfconjugate light unflavoured ones. This Lorentz covariant theory is defined via a natural generalization of the NR Lippmann-Schwinger equation and does not require further specification in the course of its solution. It does not start from the Bethe Salpeter equations. The main difference is that in the intermediate states all particles remain on their mass shell, and that total three velocity rather than four momentum is conserved. Negative energy states are not included. Retardation effects are incorporated in a simple and unambiguous way. This is to be contrasted with the Bethe-Salpeter equations, where different three-dimensional quasi-potential reductions lead to different retardations (see e.g. Sec. (2.3) of [1]). The theory has proven to give the correct fine structure formulas for the hydrogen atom and positronium [12]. In Sec. II a brief summary of the theory will be given. In Sec. III a number of quark-antiquark potentials will be discussed. A modification of the Richardson potential, to account for the OGE, as well as a relativistic generalization of the constant potential is defined. An important feature of the mesons consisting of light quarks is the appearance of linear Regge trajectories. Their origin in the light of the present theory is discussed in Sec. IV. The numerical method used will be described in Sec. V, and its results will be further discussed in Sec. VI. The paper ends with some conclusions.

II. FORMULATION OF THE THEORY

In this section a summary of the theory, as introduced by de Groot and Ruijgrok [8–11], with the modifications made in [12], will be given

A. The general framework

A state α of a quark (mass m_1) and an antiquark (mass m_2), can be characterized by $(p_1\lambda_1, p_2\lambda_2)$, where p_1 and p_2 are the four-momenta of the quark and antiquark, and λ_1 and λ_2 are their helicities. Both particles are supposed to remain on their mass shell, which means that $p_i^0 = \sqrt{|\mathbf{p}_i|^2 + m_i^2} \equiv E_i$, $i = 1, 2$. The theory is constructed in such a way that in the interaction the three-velocity

$$\mathbf{v} = \frac{\mathbf{p}_1 + \mathbf{p}_2}{p_1^0 + p_2^0}, \quad (2.1)$$

is conserved. This means that the quark-antiquark potential $V_{\beta\alpha}$ for a transition from an initial state $\alpha = (p_1\lambda_1, p_2\lambda_2)$ to a final state $\beta = (p'_1\lambda'_1, p'_2\lambda'_2)$ contains only non-zero elements if $\mathbf{v}' = \mathbf{v}$. In the center of momentum system (cms) this velocity conservation coincides with three-momentum conservation, i.e., $\mathbf{p}_1 = -\mathbf{p}_2 \equiv \mathbf{p}$ and $\mathbf{p}'_1 = -\mathbf{p}'_2 \equiv \mathbf{p}'$. In this frame therefore the potential can be written as

$$V_{\beta\alpha} = V_{\lambda'_1\lambda'_2, \lambda_1\lambda_2}(\mathbf{p}', \mathbf{p}).$$

In the NR case the momentum dependence of a central potential appears in the form $|\mathbf{q}|^2 = |\mathbf{p}' - \mathbf{p}|^2$. In the relativistic case this expression must be replaced by a covariant one. Here the usual replacement $|\mathbf{q}|^2 \rightarrow -q^2$ cannot be used because, due to the lack of four-momentum conservation, the loss of momentum $q_1 = p_1 - p'_1$ by the quark, will in general differ from the gain of momentum $q_2 = p'_2 - p_2$ by the antiquark. Instead the following obvious and symmetrical substitution is made

$$|\mathbf{q}|^2 \rightarrow -q_1 \cdot q_2 = |\mathbf{p}' - \mathbf{p}|^2 - \tau(p', p), \quad (2.2)$$

where the term τ , defined by

$$\tau(p', p) = (E_1 - E'_1)(E'_2 - E_2), \quad (2.3)$$

is responsible for retardation effects. The theoretical justification for this replacement is twofold. In the first place it can be shown [12] to be consistent with velocity conservation. The second and more practical justification is, that in the case of the Coulomb potential, τ automatically generates the correct form for the Breit interaction (see [12]). In the equal mass case $\tau = -(E - E')^2$, which is exactly opposite to the retardation used by Gross and Milana [4] and Maung, Kahana and Norbury [6]. The difference in sign will give the wrong sign for the Breit interaction, which in turn will effect the fine structure of positronium. In [7] it was shown that Eq. (2.3) gives rise to the correct positronium fine structure formula.

The relativistic wave equation from which the mass M of the meson is to be solved is in the cms given by ($\hbar = c = 1$)

$$[E_1 + E_2 - M]\Psi_{\lambda_1\lambda_2}(\mathbf{p}) + \sum_{\lambda'_1\lambda'_2} \int W_{\lambda'_1\lambda'_2,\lambda_1\lambda_2}(\mathbf{p}', \mathbf{p}) \Psi_{\lambda'_1\lambda'_2}(\mathbf{p}') \left[\frac{m_1 m_2}{E'_1 E'_2} \right] d\mathbf{p}' = 0, \quad (2.4)$$

where the wave function $\Psi_{\lambda_1\lambda_2}(\mathbf{p})$ is normalized as

$$\sum_{\lambda_1\lambda_2} \int |\Psi_{\lambda_1\lambda_2}(\mathbf{p})|^2 \left[\frac{m_1 m_2}{E_1 E_2} \right] d\mathbf{p} = 1 \quad (2.5)$$

and $V = 4m_1 m_2 W$. The quantity W is introduced for convenience, because it reduces in the NR limit to the NR potential. In this limit Eq. (2.4) reduces to the NR Schrödinger equation in momentum space.

B. Decomposition

The interaction W used in this paper can be decomposed into a vector part V_V and a scalar part V_S , which is in the cms given by

$$W(\mathbf{p}', \mathbf{p}) = \bar{u}_{\lambda'_1}(\mathbf{p}'_1) \bar{v}_{\lambda_2}(\mathbf{p}_2) \left[\gamma_\mu^{(1)} \cdot \gamma^{(2)\mu} V_V(\mathbf{p}', \mathbf{p}) + \mathbb{1}^{(1)} \mathbb{1}^{(2)} V_S(\mathbf{p}', \mathbf{p}) \right] v_{\lambda'_2}(\mathbf{p}'_2) u_{\lambda_1}(\mathbf{p}_1). \quad (2.6)$$

Here the Dirac spinors u and v for the quark resp. antiquark, are defined by

$$u_\lambda(\mathbf{p}) = N \begin{bmatrix} 1 \\ 2\lambda b \end{bmatrix} \chi(\lambda, \frac{\mathbf{p}}{p}),$$

$$v_\lambda(\mathbf{p}) = N \begin{bmatrix} -2\lambda b \\ 1 \end{bmatrix} (-i)\sigma_2 \chi^*(\lambda, \frac{\mathbf{p}}{p}),$$

with $N = \sqrt{(E+m)/(2m)}$, $b = p/(E+m)$, and $\chi(\lambda, \mathbf{p}/p)$ the helicity spinor with helicity λ . For the two-particle helicity states we use the conventions introduced by Jacob and Wick [13]. Potential (2.6) partially decouples with respect to the states $|p; JM; \lambda_1\lambda_2\rangle$, which are defined by Eq. (18) of [13], giving

$$\langle p'; J' M'; \lambda'_1 \lambda'_2 | W | p; JM; \lambda_1 \lambda_2 \rangle = \delta_{JJ'} \delta_{MM'} \langle \lambda'_1 \lambda'_2 | W^J(p', p) | \lambda_1 \lambda_2 \rangle. \quad (2.7)$$

Because of conservation of parity, W further decomposes into two 2×2 submatrices, each having a definite parity. The subspace spanned by

$$|t_1\rangle = \frac{1}{\sqrt{2}} \left[\left| \frac{1}{2}, \frac{1}{2} \right\rangle + \left| -\frac{1}{2}, -\frac{1}{2} \right\rangle \right],$$

$$|t_2\rangle = \frac{1}{\sqrt{2}} \left[\left| \frac{1}{2}, -\frac{1}{2} \right\rangle + \left| -\frac{1}{2}, \frac{1}{2} \right\rangle \right], \quad (2.8)$$

has parity $(-1)^{J+1}$. It contains the triplet $J = l \pm 1$ states. The complementary subspace, spanned by

$$\begin{aligned}
|s_1\rangle &= \frac{1}{\sqrt{2}} \left[\left| \frac{1}{2}, \frac{1}{2} \right\rangle - \left| -\frac{1}{2}, -\frac{1}{2} \right\rangle \right], \\
|s_2\rangle &= \frac{1}{\sqrt{2}} \left[\left| \frac{1}{2}, -\frac{1}{2} \right\rangle - \left| -\frac{1}{2}, \frac{1}{2} \right\rangle \right],
\end{aligned}
\tag{2.9}$$

has parity $(-1)^J$ and contains the $J = l$ singlet and triplet states. Only in the equal mass case this subspace further splits into two 1×1 subspaces. Let

$$V_{ij}^{nJ} = p' p \frac{m_1 m_2}{\sqrt{E'_1 E'_2 E_1 E_2}} \langle n_i | W^J(p', p) | n_j \rangle, \quad n = s, t, \tag{2.10}$$

then eigenvalue Eq. (2.4) can be cast in the form (suppressing the quantumnumbers J , M and s or t)

$$[E_1 + E_2 - M] f_i(p) + \sum_j \int_0^\infty V_{ij}(p', p) f_j(p') dp' = 0. \tag{2.11}$$

In appendix A explicit formula's for $V_{ij}^{nJ} = (V_V)_{ij}^{nJ} + (V_S)_{ij}^{nJ}$ are given. The reduced wave function f is normalized to

$$\sum_i \int_0^\infty |f_i(p)|^2 dp = 1. \tag{2.12}$$

III. THE QUARK-ANTIQUARK POTENTIAL

The quark-antiquark potential must contain an one-gluon exchange (OGE) to account for the short range, and a confining part for the long range interaction. It is generally believed that V_{OGE} should have a vector Lorentz structure, while about the confining part V_{CON} there is no consensus. Some believe that it must have a purely scalar structure, while others admit a mixture between scalar and vector coupling. We will adopt this last point of view. The potential can therefore be written in the form (see Eq. (2.6))

$$\begin{aligned}
V_V &= V_{\text{OGE}} + \epsilon V_{\text{CON}}, \\
V_S &= (1 - \epsilon) V_{\text{CON}},
\end{aligned}
\tag{3.1}$$

where ϵ is the scalar-vector mixing of the confining potential.

For V_{CON} the relativistic generalization of the linear potential, as described in [7], plus a constant potential (to be defined below) was used. This generalization is defined in a formal way and does not introduce any singularities. For the OGE two different potentials were used: the Richardson potential [14] and a modified version of this potential, both containing a running coupling constant (see Sec. III A). The Richardson potential contains a linear part by itself. Therefore in this case (from now on denoted by I), $\epsilon = 0$ was chosen, so that the confining in the vector direction is completely determined by the Richardson potential. The modified Richardson potential has no linear part. Therefore in this case (denoted by II) a nonzero ϵ was admitted.

In all these potentials the NR momentum transfer $|\mathbf{q}|^2$ was replaced by $-q_1 \cdot q_2$ (see Eq. (2.2)) and in this way retardation effects were included everywhere. For notational convenience, the quantity

$$Q \equiv \sqrt{(-q_1 \cdot q_2)} \quad (3.2)$$

is introduced. In the NR limit it reduces to $|\mathbf{q}|$.

A. The one-gluon-exchange: running coupling constant

The renormalization scheme of perturbative QCD says that, for large momentum transfer, the running coupling constant α_s as it occurs in the one gluon exchange (the factor $\frac{4}{3}$ arises from color averaging)

$$V_{\text{OGE}} = -\frac{4}{3} \frac{\alpha_s(Q^2)}{2\pi^2 Q^2} \quad (3.3)$$

is given by (see also Eq. (B.2) of [15])

$$\alpha_s(Q^2) = \frac{a_n}{X_n} \left[1 - b_n \frac{\log(X_n)}{X_n} + \mathcal{O}\left(\frac{\log^2(X_n)}{X_n^2}\right) \right]. \quad (3.4)$$

Here

$$a_n = \frac{12\pi}{(33 - 2n)}, \quad b_n = \frac{6(153 - 19n)}{(33 - 2n)^2},$$

$$X_n = \log \left[Q / \Lambda_{\overline{\text{MS}}}^{(n)} \right]^2,$$

and n is the number of quarks with a mass smaller than Q . The subscript $\overline{\text{MS}}$ denotes that the renormalization is performed according to the modified minimal subtraction scheme. The connection between the different $\Lambda_{\overline{\text{MS}}}^{(n)}$'s is given by Eq. (B.4) of [15]. The typical momentum transfer within a meson is on the order of one GeV, so in this region $n = 3$. Therefore, Eq. (3.4) with $n = 3$ is in many cases used as an approximation for all large momentum transfers. In addition the b_3 term is almost always neglected. But this term is not small at all: in the Q -region from 1 to 5 GeV its contribution is about 25%. Even for very high momentum transfers its contribution is substantial $\sim 15\%$ for $Q = 50$ GeV. However, it appears that when $\Lambda_{\overline{\text{MS}}}^{(5)}$ rather than $\Lambda_{\overline{\text{MS}}}^{(3)}$ is used, a fairly good approximation of Eq. (3.4) for large Q is obtained by putting

$$\alpha_s \approx \frac{a_3}{X_5} \quad (3.5)$$

(see the curve “standard approximation” of Fig. 1). For $Q = 5$ GeV the deviation from Eq. (3.4) is $\sim 7\%$, and for $Q \sim 50$ GeV there is no detectable difference. Also for smaller

Q the agreement is better, but of course in this region the validity of Eq. (3.4) is doubtful. Nevertheless we think that these considerations show that there is no theoretical necessity to stick to the value of a_3 in Eq. (3.5): a small deviation from it also results in a good running coupling constant for large Q .

For small positive Q -values Eqs. (3.4) and (3.5) diverge. To remedy this, Richardson [14] proposed a potential in which the divergence is shifted to the origin by making the replacement $Q^2 \rightarrow Q^2 + \Lambda^2$ in Eq. (3.5):

$$\begin{aligned} V_R(Q^2) &= -\frac{\alpha_0}{2\pi^2 Q^2 \log[1 + \frac{Q^2}{\Lambda^2}]} \\ &= -\frac{\alpha_0 \Lambda^2}{2\pi^2 Q^4} - \frac{\alpha_0}{4\pi^2 Q^2} + \dots \text{ for } Q \rightarrow 0. \end{aligned} \quad (3.6)$$

The color factor $\frac{4}{3}$ is absorbed in α_0 . In Fig. 1 the running coupling constant, defined via Eq. (3.3) with $V_{\text{CON}} = V_R$ is compared to the QCD formula for $\alpha_0 = \frac{4}{3}a_3 = 16\pi/27 = 1.862$. The singularity for $Q = 0$ results from a linear term in the potential with string tension $\frac{1}{2}\alpha_0\Lambda^2$ (see [7] or Sec. IIIB). When the singularity is subtracted, the running coupling constant saturates to the value $\frac{1}{2}a_3 = 0.698$. From Fig. 1 it is seen that for momentum transfers starting from 2 GeV, a much better approximation to Eq. (3.4) is obtained, if α_0 is slightly decreased. A value of $\alpha_0 = 1.750$ turns out to be a very good choice.

A different way to remove the singularities is to also make the replacement $Q^2 \rightarrow Q^2 + \Lambda^2$ in $1/Q^2$ itself. This results into

$$\begin{aligned} V_M(Q^2) &= -\frac{\alpha_0}{2\pi^2 [Q^2 + \Lambda^2] \log[1 + \frac{Q^2}{\Lambda^2}]} \\ &= -\frac{\alpha_0}{2\pi^2 Q^2} + \dots \text{ for } Q \rightarrow 0. \end{aligned} \quad (3.7)$$

This modified Richardson potential V_M does not contain a linear part. The coupling constant saturates to a value a_3 . The running coupling constant defined via this potential is given in Fig. 1 for $\alpha_0 = \frac{4}{3}a_3$. From this figure it is seen that this choice for α_0 gives a good representation of the QCD formula for moderate Q values.

The spinless partial waves W_R and W_M of V_R and V_M , defined by Eq. (A1), are given in appendix B.

B. The confining: “linear + constant” potential

For the confinement a relativistic generalization of a linear plus constant potential was used:

$$V_{\text{CON}} = V_{\text{LIN}} + V_{\text{C}}.$$

As was already mentioned in the introduction, it was for a long time believed that a linear potential could not correctly be described in momentum space. A naive consideration shows that it behaves like $-1/Q^4$, which results in an ill-defined bound state equation. A few

years ago it was shown [2,3,5] that this singularity for the NR case is only apparent. For the relativistic case different methods were employed [4,6,7] to solve this problem. The one used in this paper is defined by [7]

$$V_{\text{LIN}} = \lim_{\eta \downarrow 0} \frac{\partial^2}{\partial \eta^2} \frac{\lambda}{2\pi^2} \left[\frac{1}{Q^2 + \eta^2} \right], \quad (3.8)$$

where the color factor $\frac{4}{3}$ is absorbed in the string tension λ . In [7] it was shown that the limit exists in a distributional sense. The result was that the integral in Eq. (2.11) is replaced by

$$\oint_0^\infty \left[V_{ij}^{(nJ)}(p', p) f_j(p') + \frac{4p^2 C_{ij}(p)}{(p'^2 - p^2)^2} f_j(p) \right] dp'. \quad (3.9)$$

Here V_{ij}^{nJ} , $n = s, t$, is the naive pointwise limit obtained from Eq. (3.8). The $1/(p' - p)^2$ singularity is removed by the quantity

$$C_{ij}(p) = \lim_{p' \rightarrow p} \left[-[p' - p]^2 V_{ij}^{(nJ)}(p', p) \right]. \quad (3.10)$$

The resulting $1/(p' - p)$ singularity is handled by the principal value integral (denoted by \oint). It was shown that this subtraction is not just a trick to avoid singularities, but is really generated by Eq. (3.8).

For a confining potential that consists of a mixture of a scalar and vector Lorentz structure (see Eq. (3.1)), the pointwise limits of the spinless partial waves (see Eq. (A1)) are given by $W_V^l = \epsilon W_L^l$ and $W_S^l = (1 - \epsilon) W_L^l$, where

$$W_L^l(p', p) = \frac{\lambda}{\pi} R(p', p) \frac{Q_l'(z_0)}{p'p} \quad (3.11)$$

and also $R(p', p)$ is given in Appendix A. Here z_0 is defined by Eq. (B2) and Q_l is the Legendre function of the second kind of order l . The $1/(p' - p)^2$ singularity of W_L^l is determined by

$$-\frac{\lambda}{\pi} \frac{R(p, p)}{(p' - p)^2 - \tau(p', p)}$$

The retardation defined by Eq. (2.3) behaves around $p' \approx p$ like

$$\tau(p', p) = -\frac{p^2}{E_1 E_2} (p' - p)^2 + \dots$$

and therefore contributes to the singularity. It follows that C_{ij} is given by

$$C_{ij}(p) = \frac{\lambda}{\pi} \left[\epsilon + (1 - \epsilon) \frac{m_1 m_2}{p_1 \cdot p_2} \right] \delta_{ij}, \quad (3.12)$$

where $p_1 \cdot p_2 = E_1 E_2 + |\mathbf{p}|^2$ is the dotproduct between the four vectors p_1 and p_2 . Note that C_{ij} does not depend on J and the parity s or t . In addition it is a manifest Lorentz covariant object.

When the interaction does not contain a linear part, the integral (3.9) coincides with the integral within Eq. (2.11). This is so because then $C_{ij} = 0$ and there is no $1/(p' - p)$ singularity, which means that the principal value coincides with an ordinary integral. Therefore replacement (3.9) in combination with Eq. (3.10) can be applied to the entire interaction. In this way a nonzero value of C automatically indicates the presence of a linear term. For the Richardson potential (3.6) with a purely vector character this results in

$$(C_R)_{ij} = \frac{\alpha_0 \Lambda^2}{2\pi} \delta_{ij}, \quad (3.13)$$

which indicates a linear term with string tension $\frac{1}{2}\alpha_0 \Lambda^2$.

In analogy with the linear potential the constant potential V_C can also be defined via the Yukawa potential. In the NR case one has in configuration space

$$V_C(r) = C = \lim_{\eta \downarrow 0} \frac{\partial}{\partial \eta} \left[-\frac{C e^{-\eta r}}{r} \right].$$

Therefore in momentum space an obvious relativistic generalization is

$$V_C(Q^2) = \lim_{\eta \downarrow 0} \frac{\partial}{\partial \eta} \left[-\frac{C}{2\pi^2(Q^2 + \eta^2)} \right]. \quad (3.14)$$

Note that this expression also includes retardations, which are hidden in Q^2 (see Eq. (3.2)). Definition (3.14) has to be included in the integral of Eq. (2.11) before the limit is taken. The spinless partial wave W_η^l of this constant potential is given by

$$W_\eta^l(p', p) = -\frac{CR(p', p)}{\pi} \frac{\eta}{p'p} Q_l' \left[z_0 + \frac{\eta^2}{2p'p} \right].$$

The only term that survives the limit $\eta \downarrow 0$ is

$$\frac{CR}{\pi} \frac{\eta}{(p' - p)^2 - \tau + \eta^2} \rightarrow CR \left[\frac{E_1 E_2}{p_1 \cdot p_2} \right]^{\frac{1}{2}} \delta(p' - p) + \dots$$

For a confining potential that has both a scalar and vector part (see Eq. (2.6)) this results in

$$(V_C)_{ij}^{nJ} = C \left[\frac{p_1 \cdot p_2}{E_1 E_2} \right]^{\frac{1}{2}} \left(\epsilon + (1 - \epsilon) \frac{m_1 m_2}{p_1 \cdot p_2} \right) \delta_{ij} \delta(p' - p). \quad (3.15)$$

IV. LINEAR REGGE TRAJECTORIES

The mesons which consist of the light u , d and s quarks only, are found to lie on so-called linear Regge trajectories. This means that there are several groups of mesons, for which the mass squared for each meson within such a group, is proportional to its angular

momentum J , i.e. $M_J^2 \approx \beta J + C$. The constant C depends on the group, the Regge slope β however is about the same for all groups. Its experimental value is $\beta \approx 1.2 \text{ (GeV)}^2$. For mesons containing a heavy c or b quark, such trajectories have not been observed. This makes it plausible that linear trajectories are induced by relativistic effects. In fact, it is known [1,16,17], that the Schrödinger equation with ultrarelativistic (UR) kinematics (i.e. $2p$ instead of $p^2/(2\mu)$) for a linear potential, does indeed give rise to linear trajectories, while the (NR) Schrödinger equation does not. The slope β solely depends on the string tension λ , namely $\beta = 8\lambda$.

For the present case a similar effect is observed. It numerically appears that the (UR) limit (i.e. $m_1, m_2 \rightarrow 0$) of bound state equation (2.11) also leads to linear trajectories, with a group independent slope β . This slope however, depends on the vector part λ_V of the string tension only. In addition, the dependence is a factor $\sqrt{2}$ larger than for the relativized Schrödinger equation, namely

$$\beta \approx (8\sqrt{2})\lambda_V. \quad (4.1)$$

As can be deduced from Appendix A, the off-diagonal elements V_{12} and V_{21} of both a vector and a scalar potential vanish in the (UR) limit. Therefore Eq.(2.11) further decouples into two single equations. For the pure vector case, it reduces for the V_{11}^{tJ} channel to

$$[2p - M] f_J(p) + \oint_0^\infty \left[V^J(p', p) f_J(p') + \frac{\lambda}{\pi} f_J(p) \right] dp' = 0, \quad (4.2)$$

with

$$V^J(p', p) = \frac{2\lambda}{\pi p' p} Q'_J \left[\frac{p'^2 + p^2 + (p' - p)^2}{2p' p} \right]. \quad (4.3)$$

This equation was solved numerically using the method described in Section V. The calculated masses (in units of $\sqrt{\lambda}$) of the lowest states for each J are presented in Table I. The Schrödinger equation with (UR) dynamics is in momentum space also given by Eq. (4.2), but now with

$$V^J(p', p) = \frac{\lambda}{\pi p' p} Q'_J \left[\frac{p'^2 + p^2}{2p' p} \right]. \quad (4.4)$$

The corresponding calculated masses are also listed in Table I. They all agree with the calculations performed by Basdevant and Boukraa (see Table I of [16]).

In principle the trajectories are expected to be linear only for large values of J . However, as Table I shows, the convergence is very fast. It was found that also for moderate masses Eq. (2.11) leads to Regge trajectories with the same relation (4.1) between β and λ . The convergence, however, is then slower. When in addition a OGE term and a constant are added to the potential, relation (4.1) is affected. The change, however, is not very large. Therefore it can be concluded that, in order to obtain reasonable Regge slopes, the string tension in the vector direction λ_V should be around $\lambda_V \approx 0.1 \text{ GeV}^2$.

V. NUMERICAL METHOD

The present model embraces eigenvalue equation (2.4) in combination with a quark-antiquark potential W consisting of an one-gluon-exchange part V_{OGE} and a confining part V_{CON} . The way in which V_{OGE} and V_{CON} enter in eigenvalue equation (2.4) is given by Eqs. (2.6) and (3.1). The OGE potential is determined by two parameters α_0 and Λ and the confining potential is determined by a string tension λ and a constant C . Furthermore a parameter ϵ can be introduced to give the confining potential a mixed scalar-vector character.

The numerical solution of the model can be divided into two parts. The first one concerns the calculation of the masses of the mesons, from eigenvalue equation (2.4), given all parameters of the potential under consideration, and the quark masses. The second part is the fitting to the experimental data. The eigenvalue equation was solved by expanding the wave function into cubic Hermite splines (see [18]). The integration region $p \in [0, \infty)$ was projected onto the finite interval $x \in [0, 1]$ by $x = (p - p_0)/(p + p_0)$, where p_0 was chosen in the physical region. On this interval N equidistant spline intervals were chosen on which $2N$ spline functions were defined. The matrix elements of the resulting eigenvalue equation for the expansion coefficients only involved single integrations of the potential times a spline function. This is a major advantage of the spline method compared to the more conventional expansion techniques, where the evaluation of matrix elements involves two-dimensional integrals. The integration was performed using Gauss-Legendre quadratures. In the case where the singular point $p' = p$ was inside the region of integration, special care had to be taken. In the first place, an even number of abscissas centered around $p' = p$ was used. In that way the principal value, which occurs for the confining potential, is automatically taken care of [5,19]. Secondly, the logarithmic singularity $\sim \log(|p' - p|)$, which is induced by both the Coulomb and the confining potential, was separately handled by means of Gaussian quadratures based on a logarithmic weight function (see e.g. Table 25.7 of [20]). Another important advantage of using Hermite splines, is their small nonzero domains. Therefore on each spline interval only a few of these splines (four for the Legendre and three for the logarithmic quadrature) were needed to obtain high accuracies. The matrix equation was solved using standard techniques [21], giving the meson masses M_i .

The choice of the projection parameter p_0 and the number of intervals N , depended on the specific meson. For instance, the typical momentum transfer for the Υ mesons ($b\bar{b}$) is about 1 GeV, so $p_0 = 1$ GeV. The masses of these mesons are all known to a high precision and they are all radial levels of the $J^{PC} = 1^{--}$ channel. The Υ_v is the tenth radial state ($n = 10$). Therefore 20 spline intervals were needed to guarantee accurate results. The K_2^* however, is the only known $J^P = 2^+$ strange meson, apart from the unconfirmed $K_2^{*'}.$ Therefore $N = 8$ was sufficient to obtain reliable results. $p_0 = 0.5$ GeV was a proper choice for this meson.

The second part of the problem was to get a good fit to the experimental data. For this purpose the merit function

$$\chi^2(a_1, \dots, a_n) = \sum_i \left[\frac{M_i^{\text{the}}(a_1, \dots, a_n) - M_i^{\text{exp}}}{\sigma_i} \right]^2 \quad (5.1)$$

has to be optimized with respect to the parameters a_1, \dots, a_n . Here i labels the mesons, M_i^{exp} and M_i^{the} denote their experimental and calculated masses, and σ_i their weights. A nonlinear

regression method, based on the Levenberg-Marquardt algorithm was used to perform the fits (see section 15.5 of [21]). This method requires as input the explicit knowledge of the derivatives of the calculated masses with respect to the fit parameters. For the present complex situation this information is not known. It is only known that the derivatives of a meson mass with respect to quark masses it does not contain, is equal to zero. Therefore the derivatives were approximated in the least time consuming way by the following expression:

$$\frac{\partial M_i^{\text{the}}}{\partial a_j} \approx \frac{M_i^{\text{the}}(a_1, \dots, a_j + \Delta, \dots, a_n) - M_i^{\text{the}}(a_1, \dots, a_n)}{\Delta}. \quad (5.2)$$

In this manner all required information, e.g. M_i^{the} and $\partial M_i^{\text{the}}/\partial a_j$, is obtained by calculating all meson masses $(n + 1)$ times. A more sophisticated method would considerably increase this number. Approximation (5.2) turned out to be very effective: starting with a physically sensible set of parameters, after four or five steps convergence to an optimum was reached. The value of the parameter Δ appeared to be of minor importance. $\Delta = 0.04$ (dimensionless, in GeV, or GeV^2 , depending on the dimension of a_j) was found to be a good choice.

All mesons regarded to be established by the 1992 Particle Data Group [15] (in Table IV indicated by a “•”) were used in the fit, with the exception of the selfconjugate (i.e. Isospin 0) light unflavoured ones. For a fair description of these mesons, an annihilation interaction from initial $q\bar{q}$ states to final $q'\bar{q}'$ states should be included. Also the charmed strange D_s^* and D_{sJ} were excluded, because of the uncertainty of their quantum numbers. Furthermore the up and down quark were considered to be of equal mass. In addition the electromagnetic interactions are completely neglected. Therefore the π^0 and π^\pm , the K^0 and K^\pm , and so on will be degenerate in this picture. Because of the indistinguishability of the u and d quark, from now on such a quark will be denoted by “ u/d ”. This accumulates to a total of 52 mesons: 11 light unflavoured ($u/d\bar{u}/\bar{d}$), 11 strange (su/\bar{d}), 4 charmed (cu/\bar{d}), 2 charmed strange ($c\bar{s}$), 10 charmonia ($c\bar{c}$), 2 bottom (bu/\bar{d}) and 12 bottomia ($b\bar{b}$).

For the weight σ_i the maximum of the uncertainty dM_i^{exp} of the measured mass and the predictive power of the model, was taken. It is difficult to give an estimate for this predictive power. In the first place quark models have a phenomenological nature; there is no direct link with QCD. In the second place, the mesons are not stable particles, but in fact resonances. The decay mechanisms, which are not incorporated in this paper, could considerably effect the position of the calculated masses. This especially applies for the mesons that have decay widths of a few hundred MeV. To account for all of this, a grid size $S = 20$ MeV was introduced to give a minimum to σ . Only for bottomium ($b\bar{b}$), a grid size of 10 MeV was used, because in this system relativistic effects are less important and most states have narrow widths. Summarizing, the weights were determined by the following formula

$$\sigma_i = \text{Max} [dM_i^{\text{exp}}, S]. \quad (5.3)$$

A few exceptions to this rule were made. The pion π and the kaon K are the ground states of the $u/d\bar{u}/\bar{d}$ resp su/\bar{d} mesons. It is commonly believed, that, in order to give a fair description of these particles the mechanism of chiral symmetry breaking should be included in the model. It appeared that also the K_0^* mass was badly described by the model. This state, however, has a large decay width of ~ 300 MeV. Therefore $\sigma_\pi = 0.4$ GeV and

$\sigma_K = \sigma_{K^*} = 0.2 \text{ GeV}$ were chosen, so that these states get an insignificant weight in the fit. Another point are the ρ' and ρ'' . These states also have large decay widths ($\sim 300 \text{ MeV}$). It appeared that best results were obtained if each state was regarded to be composed of two neighboring resonances. In Sec. VI this point will be discussed in more detail.

To decrease the computation time first a rough fit was made by taking only half of the spline intervals N needed to obtain the desired accuracies. The resulting fit parameters were then used as the starting point for a full accuracy fit. The typical computation time for a complete rough fit for all 52 mesons was 30 min on a Sparc 2 workstation, while the fine tuning fit took about one hour.

As was already mentioned, two different types of potentials were examined. In case I the Richardson potential V_R was taken to account for the OGE and for the confinement in the vector direction. V_{CON} has a purely scalar character ($\epsilon = 0$). For α_0 both the “QCD” value $16\pi/27$ and the value 1.75, which gives a better agreement with the QCD formula (3.4), were taken. Both choices ended in comparable fits ($\chi^2 \approx 260$). The resulting parameters for $\alpha_0 = 1.75$ (denoted by Ia) are given in Table III and the calculated meson spectrum in Table IV. Also the case in which neither α_0 nor ϵ was fixed was regarded. The regression method led to very small values for ϵ and the string tension λ . Therefore a fit, denoted by Ib, was made where these two parameters were put equal to zero, and where α_0 was varied. This resulted in a somewhat better fit (Ib) with $\chi^2 = 250$ (see also Tables III and IV). In both cases seven parameters, three to model the potential and four quark masses, were fit. Finally, the case was considered in which the linear term of V_R was subtracted. To get a confining potential in the vector direction, the mixing ϵ was also varied. The results did not improve, however.

In case II the modified Richardson potential V_M in combination with a mixed scalar-vector V_{CON} was taken. The value $\alpha_0 = 16\pi/27$ was the only parameter held fixed, so that eight parameters were varied. In spite of the extra parameter, the resulting fit (II), see Tables III and IV, is worse than the fits found for case I and gave $\chi^2 = 322$.

VI. DISCUSSION

The meson spectrum calculated for parameter sets Ia, Ib and II is given in Table IV. Also the mesons that were not involved in the fitting procedure (the ones without a σ) were calculated. It is seen that most of these unconfirmed mesons (see [15]), are reasonably described by the model. Many states are a mixture between two $^{2s+1}L_J$ waves. Only in the NR limit these waves decouple because only then the angular momentum l is a good quantum number. For each state the most dominant wave is underlined. Most distributions are like 99% vs. 1%, which supports the statement that l is almost a good quantum number.

A few years ago (for a review, see page VII of [15]), the $\rho(1450)$ and $\rho(1700)$ were recognized as being a splitting in the formerly known $\rho(1600)$ resonance. It could be interpreted as the fine structure splitting between the $n = 2$, dominantly S , and the $n = 3$, dominantly D , states. The splitting however is rather big ($\sim 250 \text{ MeV}$). For the present model this interpretation was found to be in conflict with the rest of the spectrum. A correct $\rho' - \rho''$ splitting induced a far too large splitting in the 1^{--} states of charmonium and bottonium, and visa versa. Only if the ρ' was regarded to consist of the $n = 2$ and $n = 3$ states, and

the ρ'' of the $n = 4$ and $n = 5$ states, correct splittings for the entire spectrum could be obtained. In addition, the correct splitting between the observed ρ''' and ρ^{iv} (~ 50 MeV) was obtained. The difference between the $n = 2$ and $n = 3$ mass, and between the $n = 4$ and $n = 5$ mass, was found to be ~ 100 MeV, which is much smaller than the decay widths of the ρ' and the ρ'' (~ 300 MeV).

The masses of the π , K and K_0^* are found to be much too high. As was already mentioned, this was to be expected, because the small masses of these particles are believed to be a consequence of spontaneous breaking of chiral symmetry. In all cases Ia, Ib and II, the mass of the η_c is found to be too high. As was pointed out by Hirono et.al. [22], this is probably a consequence of neglecting negative energy states. They found that for a quark model based on the instantaneous ladder BS equation for charmonium, the η_c is strongly influenced by neglecting these states (~ 100 MeV), while the dependence on all other states is much weaker (~ 10 MeV). If one extrapolates these results to the present theory, this means that the omission of the negative energy states only weakly affects the spectrum. Only for the 1S_0 ground states a substantial mass drop may arise. This would mean that also the masses of the D and the D_S , which are now found a bit too high, would become smaller. The B would also get a smaller mass, which only has a positive result in case Ia.

The centre of gravity COG(n) (see e.g. Sec. 8.1 of [1]) is defined by:

$$\text{COG}(n) \equiv \frac{5}{9}M(n^3P_2) + \frac{1}{3}M(n^3P_1) + \frac{1}{9}M(n^3P_0)$$

It can be proved that, for an arbitrary scalar potential V_S and a Coulomb vector potential V_V , up to first order relativistic corrections, this COG equals the corresponding n^1P_1 singlet. The relation is violated by the Q -dependence of α_s and the presence of a confining term in the vector direction. It is also affected by higher order relativistic corrections. In all cases the COG is found to be somewhat higher than the corresponding singlet state. A related quantity is the ratio [1,24]

$$\rho = \frac{M(^3P_2) - M(^3P_1)}{M(^3P_1) - M(^3P_0)}. \quad (6.1)$$

Its experimental value is 0.21 for $u/d\overline{u}/\overline{d}$, 0.48 for $c\overline{c}$, 0.66 for $n = 1$ $b\overline{b}$ and 0.57 for $n = 2$ $b\overline{b}$. For all three cases Ia, Ib and II, a rather constant value of $\rho \sim 0.8$ (see Table III) was found. A perturbative configuration space calculation shows (see e.g. Sec. 4.2 of [1]) that this too large value for ρ is a consequence of the dominance of the vector OGE. An analysis for the present case in momentum space, gives a similar result. A more profound linear scalar potential might lower this ratio.

The following remarks on the parameter sets can be made. From Table III it is seen that the quark masses are substantially larger than usual in quark models. Furthermore, the masses are quite different for the different cases. The smallest masses are obtained by fit Ia. This is a consequence of the large negative constant $C \sim -1.0$ GeV, which, however, is necessary in order to obtain a good fit for the entire spectrum. If, for instance one only considers bottomonium and charmonium, it turns out that the quality of the fit only weakly depends on the value of C . The system is overparametrized and one in fact does not even need a constant in the potential. But, when simultaneously also good results for the lighter mesons are required, the large negative constant arises automatically.

The total string tension λ_{tot} is defined as the sum of the tensions in vector and scalar direction. For case I one has $\lambda_{\text{tot}} = \lambda + \frac{1}{2}\alpha_0\Lambda^2$, while case II simply gives $\lambda_{\text{tot}} = \lambda$. These tensions are also quite different for the different cases. In case Ib, which gave the best fit, there is only a vector tension. This is in contrast to the requirement that, in order to obtain better ρ values, the confining should be dominantly scalar. The total tension for case Ia is closest to value $\lambda \sim 0.18 \text{ GeV}^2$, which is often given in the literature.

The Regge slopes of Ia and Ib are compatible with the experimental value $\beta \approx 1.2 \text{ GeV}^2$. The slope found in II, is somewhat too low. This can clearly be seen from the high- J states like the ρ_5 , a_6 and K_5^* . The errors given in Table III represent a measure of linearity of the trajectories. It is defined as the spread in the difference between the masses squared of adjacent states. The spreads found are considerably smaller than the experimental value.

Finally the running coupling constant for $Q = 31 \text{ GeV}$ and $Q = M_Z = 91.16 \text{ GeV}$ were compared with the experimental values

$$\begin{aligned}\alpha_s(34 \text{ GeV}) &= 0.14 \pm 0.02, \\ \alpha_s(M_Z) &= 0.1134 \pm 0.0035.\end{aligned}$$

Only case Ia is compatible with both conditions. The choice $\alpha_0 = 1.75$ was made to give the best approximation to the QCD formula (3.4) for moderate momentum transfer. Now it appears that this choice also gives correct results for very high momenta. A fit of type Ia, but now with $\alpha_0 = 16\pi/27$ (not displayed) gave a too large high momentum α_s . In principle, the high Q -range of the potential is completely irrelevant for the calculation of the meson spectrum, where the potential is only tested up to a few GeV. Nevertheless, for the sake of theoretical consistency, this test was made.

VII. CONCLUDING REMARKS

In this paper a relativistic quark model defined in momentum space was studied. The quark-antiquark potential used, consisted of a OGE with a Lorentz vector character, and a linear plus constant confining potential. For the OGE the Richardson potential V_R , given by Eq. (3.6), with and without its linear part, as well a modified Richardson potential V_M , defined by Eq. (3.7), was regarded. Best results were obtained for the Richardson potential including its linear term (case I). The linear plus constant potential was given a pure scalar character, i.e. $\epsilon = 0$ in Eq. (2.6). In this way, the confining in the vector direction was induced by the linear part of V_R . For case I two different fits were made, fit Ia, in which the value of α_0 was fixed to 1.75, and fit Ib, in which α_0 was varied, but the string tension in the scalar direction λ was put equal to zero. Also reasonable results were obtained for $V_{\text{OGE}} = V_R$ (case II). Here the confining potential was given a mixed scalar-vector character. For the fits Ia, Ib and II, most meson masses, with the exception of the π , K and K_0^* were found to be reasonably described by the model. In case Ia and Ib correct Regge slopes were found, and only in case Ia a correct strong coupling constant for large momenta was found. The ratios ρ , defined by Eq. (6.1), however, were in all three cases found to be too large. It is concluded that case Ia should be preferred.

No detailed comparison with other theories has been made, because the main purpose of this paper was not so much to improve upon the existing calculations, but rather to

show that results of the same quality could be obtained using a relativistic theory which is formulated in the momentum representation.

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APPENDIX A: PARTIAL WAVE DECOMPOSITION

In this appendix we give the precise form of the decomposition of a potential W defined by Eq. (2.6). The partial waves,

$$V_{ij}^{nJ} = (V_V)_{ij}^{nJ} + (V_S)_{ij}^{nJ}, \quad n = s, t,$$

defined by Eq. (2.10) can in general be expressed in terms of the “spinless” partial waves W_V^l and W_S^l of V_V and V_S respectively. They are defined by

$$W_{V,S}^l(p', p) = (2\pi p' p) R(p', p) \int_{-1}^{+1} W_{V,S}(\mathbf{p}', \mathbf{p}) P_l(x) dx, \quad (\text{A1})$$

with $x = \frac{\mathbf{p}' \cdot \mathbf{p}}{p' p}$ and P_l the Legendre polynomial of order l . The quantity R is defined by $R(p', p) = A_1' A_2' A_1 A_2$, with $A = \sqrt{\frac{E+m}{2E}}$. If furthermore $b = \frac{p}{E+m}$, then the result is:

1. Vectorpotential for the $|s_{1,2}\rangle$ states

$$\begin{aligned} (V_V)_{11}^{sJ}(p', p) &= [1 + 3(b_1' b_2' + b_1 b_2) + b_1' b_2' b_1 b_2] W_V^J + (b_1' - b_2')(b_1 - b_2) \frac{(J+1)W_V^{J+1} + J W_V^{J-1}}{2J+1}, \\ (V_V)_{12}^{sJ}(p', p) &= (b_2' - b_1')(b_2 + b_1) \sqrt{J(J+1)} \frac{W_V^{J+1} - W_V^{J-1}}{2J+1} = (V_V)_{21}^{sJ}(p, p'), \\ (V_V)_{22}^{sJ}(p', p) &= (1 + b_1' b_2')(1 + b_1 b_2) W_V^J + (b_1' + b_2')(b_1 + b_2) \frac{J W_V^{J+1} + (J+1) W_V^{J-1}}{2J+1}. \end{aligned} \quad (\text{A2})$$

2. Vectorpotential for the $|t_{1,2}\rangle$ states

$$\begin{aligned} (V_V)_{11}^{tJ}(p', p) &= (1 - b_1' b_2')(1 - b_1 b_2) \frac{(J+1)W_V^{J+1} + J W_V^{J-1}}{2J+1} \\ &\quad + [(b_1' - b_2')(b_1 - b_2) + 4(b_1' b_2 + b_2' b_1)] W_V^J, \\ (V_V)_{12}^{tJ}(p', p) &= -(1 - b_1' b_2')(1 + b_1 b_2) \sqrt{J(J+1)} \frac{W_V^{J+1} - W_V^{J-1}}{2J+1} = (V_V)_{21}^{tJ}(p, p'), \\ (V_V)_{22}^{tJ}(p', p) &= (1 + b_1' b_2')(1 + b_1 b_2) \frac{J W_V^{J+1} + (J+1) W_V^{J-1}}{2J+1} + (b_1' + b_2')(b_1 + b_2) W_V^J. \end{aligned} \quad (\text{A3})$$

3. Scalarpotential for the $|s_{1,2}\rangle$ states

$$\begin{aligned}
(V_S)^{sJ}_{11}(p', p) &= (1 + b'_1 b'_2 b_1 b_2) W_S^J - (b'_1 b_1 + b'_2 b_2) \frac{(J+1)W_S^{J+1} + JW_S^{J-1}}{2J+1}, \\
(V_S)^{sJ}_{12}(p', p) &= (b'_1 b_1 - b'_2 b_2) \sqrt{J(J+1)} \frac{W_S^{J+1} - W_S^{J-1}}{2J+1} = (V_S)^{sJ}_{21}(p, p'), \\
(V_S)^{sJ}_{22}(p', p) &= (1 + b'_1 b'_2 b_1 b_2) W_S^J - (b'_1 b_1 + b'_2 b_2) \frac{JW_S^{J+1} + (J+1)W_S^{J-1}}{2J+1}.
\end{aligned} \tag{A4}$$

4. Scalarpotential for the $|t_{1,2}\rangle$ states

$$\begin{aligned}
(V_S)^{tJ}_{11}(p', p) &= (1 + b'_1 b'_2 b_1 b_2) \frac{(J+1)W_S^{J+1} + JW_S^{J-1}}{2J+1} - (b'_1 b_1 + b'_2 b_2) W_S^J, \\
(V_S)^{tJ}_{12}(p', p) &= -(1 - b'_1 b'_2 b_1 b_2) \sqrt{J(J+1)} \frac{W_S^{J+1} - W_S^{J-1}}{2J+1} = (V_S)^{tJ}_{21}(p, p'), \\
(V_S)^{tJ}_{22}(p', p) &= (1 + b'_1 b'_2 b_1 b_2) \frac{JW_S^{J+1} + (J+1)W_S^{J-1}}{2J+1} - (b'_1 b_1 + b'_2 b_2) W_S^J.
\end{aligned} \tag{A5}$$

Strictly speaking, these results are only valid for $J > 0$. For $J = 0$ only the V_{11} 's are nonzero and are also given by Eqs. (A2,...,A5), but with $W^{l=-1} = 0$.

In the equal mass case there is no difference between the b_1 's and the b_2 's. From this it is seen that the V_{12}^{sJ} 's and the V_{21}^{sJ} 's are zero. This means that the potential decouples with regard to the $|s_1\rangle$ state, which corresponds to the $l = J$ singlet, and the $|s_2\rangle$ state, which corresponds to the $l = J$ triplet. Therefore in this case only the $|l = J \pm 1\rangle$ triplet states mix. In the unequal mass case the $l = J$ singlet and triplet states mix.

APPENDIX B: SPINLESS DECOMPOSITION OF THE RICHARDSON AND MODIFIED RICHARDSON POTENTIAL

In this appendix the spinless partial wave decomposition W_R and W_M of the potentials V_R and V_M , defined by Eqs. (3.6) and (3.7) will be calculated.

The momentum transfer Q^2 which is defined by Eq. (3.2) depends on the lengths $p = |\mathbf{p}|$ and $p' = |\mathbf{p}'|$ of the incoming resp. outgoing momentum, and on the angle $x = \frac{\mathbf{p} \cdot \mathbf{p}'}{pp'}$ between these two momenta:

$$Q^2 = Q^2(p, p', x) = 2pp'(z_0 - x). \tag{B1}$$

Here

$$z_0(p, p') = \frac{p^2 + p'^2 - \tau(p, p')}{2pp'}, \tag{B2}$$

where the retardation τ is a theory dependent quantity which in the present case is given by Eq. (2.3) The spinless partial wave $W^l(p', p)$ corresponding to an angular momentum l is defined by Eq. (A1). Introducing

$$y(x) = 1 + \frac{Q^2(x)}{\Lambda^2}, \quad b = \frac{\Lambda^2}{2pp'} \quad (\text{B3})$$

and $y_{\pm} = y(x = \mp 1) \geq 1$, then W_R^l and W_M^l are given by

$$W_R^l = -\frac{\alpha_0 R}{2\pi} \int_{y_-}^{y_+} \frac{P_l[z_0 - b(y-1)]}{(y-1) \log y} dy, \quad (\text{B4})$$

$$W_M^l = -\frac{\alpha_0 R}{2\pi} \int_{y_-}^{y_+} \frac{P_l[z_0 - b(y-1)]}{y \log y} dy. \quad (\text{B5})$$

The y dependence in P_l can be expanded, using

$$P_l(z-w) = \sum_{i=0}^l g_i^l(z) w^i. \quad (\text{B6})$$

Here g_i^l is a polynomial of degree $l-i$. For $l \leq 3$ it is given in Table I. For general l it can be found from the recurrence relation

$$(l+1)g_i^{l+1} = (2l+1)(zg_i^l - g_{i-1}^l) - lg_i^{l-1}, \quad (\text{B7})$$

in combination with the initial values

$$g_0^{-1}(z) = 0, \quad g_0^0(z) = 1. \quad (\text{B8})$$

Note that g_0^l obeys the recurrence relation of the Legendre Polynomials. In combination with the initial values (B8) it follows that $g_0^l = P_l$.

The partial waves W_R^l and W_M^l can be written in terms of g_i^l 's and the integrals

$$A_n = \frac{b^n}{2} \int_{y_-}^{y_+} \frac{(y-1)^n}{y \log y} dy, \quad n \geq -1. \quad (\text{B9})$$

For $n = -1$ one has

$$A_{-1} = \frac{1}{2b} [F(\log y_+) - F(\log y_-)], \quad (\text{B10})$$

where

$$F(x) \equiv - \int_x^\infty \frac{dt}{t(e^t - 1)}, \quad x > 0. \quad (\text{B11})$$

For $n \geq 0$ the integrals A_n can be expanded into

$$A_n = \frac{b^n}{2} \sum_{k=0}^n \binom{n}{k} (-1)^{(n-k)} I_k, \quad (\text{B12})$$

where

$$I_n = \int_{y_-}^{y_+} \frac{y^{n-1}}{\log y} dy, \quad n \geq 0. \quad (\text{B13})$$

One finds

$$I_0 = \log \left[\frac{\log y_+}{\log y_-} \right], \quad (\text{B14})$$

$$I_n = Ei(n \log y_+) - Ei(n \log y_-), \quad n > 0, \quad (\text{B15})$$

where

$$Ei(x) \equiv \int_{-\infty}^x \frac{e^t}{t} dt \quad (\text{B16})$$

is the Exponential integral (see Eq. (5.1.2) of [20]). The principal value integral is denoted by \mathcal{P} .

Summarizing all steps it follows that W_R^l and W_M^l , defined by Eqs. (A1), (3.6) and (3.7) are given by

$$W_R^l(p, p') = -\frac{\alpha_0 R}{\pi} \sum_{i=0}^l g_i^l(z_0) (A_i + b A_{i-1}), \quad (\text{B17})$$

$$W_M^l(p, p') = -\frac{\alpha_0 R}{\pi} \sum_{i=0}^l g_i^l(z_0) A_i, \quad (\text{B18})$$

where the polynomials g_i^l are defined by Eq. (B6) and the integrals A_n can be found from Eqs. (B10) and (B12).

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FIGURES

FIG. 1. Running coupling constant $\alpha_s(Q^2)$, defined by Eq. (3.3) for three different choices of V_{OGE} compared to the QCD formula (3.4) and its standard approximation (3.5). $\Lambda = \Lambda_{\overline{MS}}^{(5)} = 0.3 \text{ GeV}$ and $\alpha_0 = 16\pi/27$.

TABLES

TABLE I. Regge trajectories calculated from the ultrarelativistic Eq. (4.2). The masses are expressed in terms of $\sqrt{\lambda}$, where λ is the string tension.

l	Present case: potential (4.3)		Basdevant and Boukra: potential (4.4)	
	M_l	$(M_l^2 - M_{l-1}^2)/(8\sqrt{2})$	M_l	$(M_l^2 - M_{l-1}^2)/8$
0	3.830		3.157	
1	5.062	0.969	4.225	0.985
2	6.066	0.987	5.079	0.994
3	6.931	0.993	5.811	0.996
4	7.701	0.996	6.461	0.998
5	8.402	0.998	7.052	0.999
6	9.049	0.998	7.597	0.998

TABLE II. The polynomials g_i^l for $l \leq 3$, defined by Eq. (B6)

$l \setminus i$	0	1	2	3
0	1			
1	z	-1		
2	$\frac{3}{2}z^2 - \frac{1}{2}$	$-3z$	$\frac{3}{2}$	
3	$\frac{5}{2}z^3 - \frac{3}{2}z$	$-\frac{15}{2}z^2 + \frac{3}{2}$	$\frac{15}{2}z$	$-\frac{5}{2}$

TABLE III. Final parameter sets from the fitting procedure described in Sec. V for potential models I and II. The varied parameters are indicated by a “•”. For model I two different fits were made. In case Ia α_0 was held fixed and λ was fitted, while in case Ib λ was put equal to 0 and α_0 was fitted. The related quantities are discussed in Sec. VI.

Model:	Ia	Ib	II
<u>Potential</u>			
α_0	1.750	2.434 •	1.862
Λ (GeV)	0.324 •	0.320 •	0.376 •
λ (GeV ²)	0.077 •	0	0.136 •
C (GeV)	-1.297 •	-1.291 •	-1.038 •
ϵ	0	0	0.523 •
<u>Quark masses</u>			
$m_{u/d}$ (GeV)	0.512 •	0.699 •	0.966 •
m_s (GeV)	0.766 •	0.889 •	1.072 •
m_c (GeV)	2.066 •	2.206 •	2.249 •
m_b (GeV)	5.474 •	5.616 •	5.593 •
# parameters	7	7	8
χ^2	263	250	322
<u>Related quantities</u>			
λ_{tot} (GeV ²)	0.169	0.125	1.136
β (GeV ²)	1.18±0.05	1.27±0.09	0.93±0.08
ρ	0.81	0.79	0.75
$\alpha_s(34\text{GeV})$	0.141	0.196	0.155
$\alpha_s(M_Z)$	0.1164	0.161	0.127

TABLE IV. Meson spectrum calculated from Eq. (2.4) for three different parameter sets Ia, Ib and II, (see Table III). All masses are in (GeV). The experimental values are taken from [15], with the exception of the h_{c1} , which is taken from [23]. The mesons labeled with a “•” (regarded as being established by [15]) were, with the exclusion of the D_s^* and the D_{sJ} , involved in the fitting procedure. The weights σ_i are determined by Eq. (5.3). The most dominant $^{2s+1}L_J$ waves are underlined.

Light unflavoured mesons: u/d quarks.								
Name i	J^{PC}	$^{2s+1}L_J$	M_i^{exp}	n	M_i^{Ia}	M_i^{Ib}	M_i^{II}	σ_i
• π	0^{-+}	1S_0	0.135	1	0.600	0.595	0.688	0.400
• π'	0^{-+}	1S_0	1.300	2	1.243	1.206	1.292	0.100
π''	0^{-+}	1S_0	1.775	3	1.711	1.671	1.695	
• ρ	1^{--}	$^3\underline{S}_1/^3D_1$	0.768	1	0.754	0.762	0.867	0.020
	1^{--}	$^3\underline{S}_1/^3D_1$		2	1.365	1.345	1.387	
• ρ'	1^{--}	$^3S_1/^3\underline{D}_1$	1.465	3	1.474	1.477	1.460	0.025
• ρ''	1^{--}	$^3\underline{S}_1/^3D_1$	1.700	4	1.806	1.786	1.764	0.020
	1^{--}	$^3S_1/^3\underline{D}_1$		5	1.865	1.864	1.807	
ρ'''	1^{--}	$^3\underline{S}_1/^3D_1$	2.100	6	2.162	2.151	2.065	
ρ^{iv}	1^{--}	$^3S_1/^3\underline{D}_1$	2.150	7	2.200	2.206	2.096	
• a_0	0^{++}	3P_0	0.983	1	1.012	0.981	1.017	0.020
a'_0	0^{++}	3P_0	1.320	2	1.517	1.464	1.510	
• a_1	1^{++}	3P_1	1.260	1	1.166	1.163	1.197	0.030
• a_2	2^{++}	$^3\underline{P}_2/^3F_2$	1.318	1	1.301	1.319	1.329	0.020
COG		$^3P_{0,1,2}$	1.262	1	1.224	1.229	1.250	
• b_1	1^{+-}	1P_1	1.232	1	1.183	1.194	1.231	0.020
• π_2	2^{-+}	1D_2	1.670	1	1.590	1.614	1.561	0.020
π'_2	2^{-+}	1D_2	2.100	2	1.958	1.986	1.880	
• ρ_3	3^{--}	$^3\underline{D}_3/^3G_3$	1.691	1	1.698	1.734	1.637	0.020
ρ'_3	3^{--}	$^3\underline{D}_3/^3G_3$	2.250	2	2.051	2.092	1.940	
	3^{--}	$^3D_3/^3\underline{G}_3$		3	2.097	2.152	1.969	
a_3	3^{++}	3F_3	2.050	1	1.915	1.957	1.813	
a_4	4^{++}	$^3\underline{F}_4/^3H_4$	2.040	1	2.021	2.085	1.883	
ρ_5	5^{--}	$^3\underline{G}_5/^3I_5$	2.350	1	2.297	2.395	2.093	
a_6	6^{++}	$^3\underline{H}_6/^3J_6$	2.450	1	2.540	2.677	2.279	

Strange mesons (Kaons): $s, u/d$ quarks.

Name i	J^{PC}	$^{2s+1}L_J$	M_i^{exp}	n	M_i^{Ia}	M_i^{Ib}	M_i^{II}	σ_i
• K	0^{-}	1S_0	0.495	1	0.762	0.723	0.781	0.200
K'	0^{-}	1S_0	1.460	2	1.402	1.329	1.385	
K''	0^{-}	1S_0	1.830	3	1.864	1.786	1.786	
• K^*	1^{-}	$^3\underline{S}_1/^3D_1$	0.894	1	0.891	0.876	0.955	0.020
• $K^{*'} $	1^{-}	$^3\underline{S}_1/^3D_1$	1.412	2	1.504	1.455	1.477	0.020
• $K^{*''}$	1^{-}	$^3S_1/^3\underline{D}_1$	1.714	3	1.618	1.588	1.553	0.020
	1^{-}	$^3\underline{S}_1/^3D_1$		4	1.945	1.890	1.852	
• K_0^*	0^{+}	3P_0	1.429	1	1.177	1.112	1.115	0.200

	0^+	3P_0		2	1.674	1.587	1.604	
$K_0^{*'} $	0^+	3P_0	1.950	3	2.074	1.989	1.955	
• K_1	1^+	$^1\underline{P}_1/^3P_1$	1.270	1	1.304	1.274	1.288	0.020
• K_1'	1^+	$^1P_1/^3\underline{P}_1$	1.402	2	1.322	1.306	1.321	0.020
K_1''	1^+	$^1\underline{P}_1/^3P_1$	1.650	3	1.773	1.725	1.701	
• K_2^*	2^+	$^3\underline{P}_2/^3F_2$	1.429	1	1.416	1.415	1.415	0.020
$K_2^{*'} $	2^+	$^3\underline{P}_2/^3F_2$	1.980	2	1.867	1.849	1.785	
	2^+	$^3P_2/^3\underline{F}_2$		3	1.945	1.934	1.831	
K_2	2^-	$^1\underline{D}_2/^3D_2$	1.580	1	1.706	1.693	1.640	
• K_2'	2^-	$^1D_2/^3\underline{D}_2$	1.768	2	1.715	1.711	1.650	0.020
K_2''	2^-	$^1\underline{D}_2/^3D_2$	2.250	3	2.082	2.065	1.962	
• K_3^*	3^-	$^3D_3/^3G_3$	1.770	1	1.801	1.813	1.722	0.020
K_3	3^+	$^1\underline{F}_3/^3F_3$	2.320	1	2.027	2.034	1.900	
• K_4^*	4^+	$^3\underline{F}_4/^3H_4$	2.045	1	2.115	2.150	1.967	0.020
K_4	4^-	$^1G_4/^3\underline{G}_4$	2.500	1	2.300	2.333	2.116	
K_5^*	5^-	$^3G_5/^3\underline{I}_5$	2.380	1	2.386	2.449	2.176	

Charmed mesons: c , u/d quarks.

Name i	J^{PC}	$^{2s+1}L_J$	M_i^{exp}	n	M_i^{Ia}	M_i^{Ib}	M_i^{II}	σ_i
• D	0^-	1S_0	1.867	1	1.935	1.901	1.904	0.020
• D^*	1^-	$^3\underline{S}_1/^3D_1$	2.010	1	2.006	1.999	2.031	0.020
• D_1	1^+	$^1\underline{P}_1/^3P_1$	2.424	1	2.406	2.379	2.382	0.020
$D_J(?)$	1^+	$^1P_1/^3\underline{P}_1$	2.440	2	2.439	2.438	2.424	
• D_2^*	2^+	$^3\underline{P}_2/^3F_2$	2.459	1	2.485	2.492	2.484	0.020

Charmed strange mesons: c , s quarks.

Name i	J^{PC}	$^{2s+1}L_J$	M_i^{exp}	n	M_i^{Ia}	M_i^{Ib}	M_i^{II}	σ_i
• D_s	0^-	1S_0	1.969	1	2.032	1.990	1.984	0.020
• $D_s^*(?)$	1^-	$^3\underline{S}_1/^3D_1$	2.110	1	2.100	2.088	2.110	
• D_{s1}	1^+	$^1\underline{P}_1/^3P_1$	2.537	1	2.498	2.473	2.466	0.020
	1^+	$^1P_1/^3\underline{P}_1$		2	2.520	2.516	2.503	
• $D_{sJ}(?)$	2^+	$^3\underline{P}_2/^3F_2$	2.564	1	2.561	2.568	2.563	

Bottom mesons: b , u/d quarks.

Name i	J^{PC}	$^{2s+1}L_J$	M_i^{exp}	n	M_i^{Ia}	M_i^{Ib}	M_i^{II}	σ_i
• B	0^-	1S_0	5.279	1	5.303	5.268	5.247	0.020
• B^*	1^-	$^3\underline{S}_1/^3D_1$	5.325	1	5.336	5.318	5.316	0.020

Charmonium: c quarks.

Name i	J^{PC}	$^{2s+1}L_J$	M_i^{exp}	n	M_i^{Ia}	M_i^{Ib}	M_i^{II}	σ_i
• η_c	0^{-+}	1S_0	2.979	1	3.042	3.010	3.007	0.020
η_c'	0^{-+}	1S_0	3.590	2	3.615	3.589	3.609	
• J/Ψ	1^{--}	$^3\underline{S}_1/^3D_1$	3.097	1	3.099	3.104	3.117	0.020
• Ψ'	1^{--}	$^3\underline{S}_1/^3D_1$	3.686	2	3.655	3.646	3.665	0.020

•	Ψ''	1^{--}	$^3S_1/^3\underline{D}_1$	3.770	3	3.766	3.780	3.775	0.020
•	Ψ'''	1^{--}	$^3\underline{S}_1/^3D_1$	4.040	4	4.051	4.017	4.028	0.020
•	Ψ^{iv}	1^{--}	$^3S_1/^3\underline{D}_1$	4.159	5	4.124	4.105	4.097	0.020
•	Ψ^v	1^{--}	$^3\underline{S}_1/^3D_1$	3.415	6	4.376	4.319	4.314	0.020
		1^{--}	$^3S_1/^3\underline{D}_1$		7	4.430	4.384	4.364	
•	χ_{c0}	0^{++}	3P_0	3.415	1	3.437	3.433	3.409	0.020
•	χ_{c1}	1^{++}	3P_1	3.511	1	3.485	3.506	3.504	0.020
•	χ_{c2}	2^{++}	$^3\underline{P}_2/^3F_2$	3.556	1	3.523	3.562	3.572	0.020
	COG		$^3P_{0,1,2}$	3.525	1	3.501	3.529	3.531	
	h_{c1}	1^{+-}	1P_1	3.526	1	3.492	3.520	3.522	

Bottonium: b quarks.

Name i	J^{PC}	$^{2s+1}L_J$	M_i^{exp}	n	M_i^{Ia}	M_i^{Ib}	M_i^{II}	σ_i
• Υ	1^{--}	$^3\underline{S}_1/^3D_1$	9.460	1	9.493	9.434	9.441	0.010
• Υ'	1^{--}	$^3\underline{S}_1/^3D_1$	10.023	2	10.011	10.018	10.022	0.010
	1^{--}	$^3S_1/^3\underline{D}_1$		3	10.131	10.171	10.160	
• Υ''	1^{--}	$^3\underline{S}_1/^3D_1$	10.355	4	10.346	10.348	10.365	0.010
	1^{--}	$^3S_1/^3\underline{D}_1$		5	10.423	10.444	10.451	
• Υ'''	1^{--}	$^3\underline{S}_1/^3D_1$	10.580	6	10.614	10.599	10.626	0.010
	1^{--}	$^3S_1/^3\underline{D}_1$		7	10.672	10.670	10.688	
• Υ^{iv}	1^{--}	$^3\underline{S}_1/^3D_1$	10.865	8	10.846	10.811	10.844	0.010
	1^{--}	$^3S_1/^3\underline{D}_1$		9	10.893	10.868	10.892	
• Υ^v	1^{--}	$^3\underline{S}_1/^3D_1$	11.019	10	11.054	11.000	11.035	0.010
• χ_{b0}	0^{++}	3P_0	9.860	1	9.859	9.863	9.843	0.010
• χ_{b0}	0^{++}	3P_0	10.232	2	10.220	10.229	10.232	0.010
• χ_{b1}	1^{++}	3P_1	9.892	1	9.882	9.906	9.888	0.010
• χ_{b1}	1^{++}	3P_1	10.255	2	10.239	10.258	10.261	0.010
• χ_{b2}	2^{++}	$^3\underline{P}_2/^3F_2$	9.913	1	9.901	9.938	9.922	0.010
• χ_{b2}	2^{++}	$^3\underline{P}_2/^3F_2$	10.268	2	10.253	10.281	10.284	0.010
	COG	$^3P_{0,1,2}$	9.900	1	9.890	9.919	9.902	
	COG	$^3P_{0,1,2}$	10.260	2	10.245	10.267	10.271	